The Basic Neoclassical Model

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Contents

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1 The Neoclassical Growth Model

1.1 Economic Environment

Preferences

The economy is populated by many identical infinitely-lived individuals with preferences over goods and leisure represented by

$$U = \sum_{t=0}^{\infty} \beta^t u(C_t, L_t) \ , \quad 0 < \beta < 1 \tag{1}$$

where C_t is commodity consumption in period t and L_t is leisure in period t. Each household has a time endowment of one unit, which can be allocated for leisure L_t or work N_t. $u(C, L)$ is the momentary utility function or felicity function. Assume that $u_c > 0$ and $u_l > 0$ and that $u(C, L)$ is strictly concave in both arguments, twice continuously differentiable and satisfies the Inada conditions:

$$\lim_{c \to 0} u_c(C, L) = \infty \qquad \lim_{c \to \infty} u_c(C, L) = 0$$
$$\lim_{L \to 0} u_l(C, L) = \infty \qquad \lim_{L \to 1} u_l(C, L) = 0$$

The parameter $0 < \beta < 1$ is the discount factor and $\frac{1-\beta}{\beta}$ is the discount rate.

Production Possibilities

There is only one final good in the economy that is produced according to the constant returns to scale neoclassical production technology given by

$$Y_t = F(K_t, N_t)$$

where K_t is the per capita predetermined capital stock and N_t is the per capita labor input in period t. "Predetermined" means that the capital stock is given in period t and is decided upon in period $t-1$. Assume that the function $F(K, N)$ is concave in both arguments, twice continuously differentiable and satisfies the Inada Conditions:

$$\lim_{K \to 0} F_k(K, N) = \infty \qquad \lim_{K \to \infty} F_k(K, N) = 0$$
$$\lim_{N \to 0} F_n(K, N) = \infty \qquad \lim_{N \to \infty} F_n(K, N) = 0$$

Capital Accumulation

The final good can be either consumed or used for investment, i.e. it can be added to the capital stock. The capital stock evolves according to

$$K_{t+1} = I_t + (1-\delta)K_t \ , \quad 0 < \delta < 1 \tag{2}$$

where I_t is per capita gross investment, $I_t - \delta K_t$ is net investment and the parameter $0 < \delta < 1$ denotes the rate of capital depreciation.

Resource Constraints

The size of the population is constant over time. In each period t, the choices of consumption, investment and factor inputs are subject to the following resource constraints, expressed in per capita terms:

$$L_t + N_t \leq 1$$
$$C_t + I_t \leq Y_t$$

The first constraint states that the time allocated to work and leisure cannot exceed the time endowment of one unit. The second constraint requires the total uses of the final good, consumption and investment, not to exceed output. In addition, there are the nonnegativity constraints: $L_t, N_t, K_{t+1}, C_t \geq 0$.

1.2 The Social Planner's Outcome

Suppose that both the consumption and production decision are made by the same economic agent, which we will call the benevolent *social planner*. The social planner's goal is to maximize the weighted average of household's utilities. Since all households have identical preferences, the objective of the social planner is to choose sequences $\{C_t, L_t, Y_t, I_t, K_{t+1}, N_t\}_{t=0}^{\infty}$ that maximize the utility function (1) subject to the technological, resource and nonnegativity constraints and taking the initial (period 0) capital stock K_0 as given. Formally, the planner's problem is

$$
\begin{aligned}
\max_{\{C_t,L_t,Y_t,I_t,K_{t+1},N_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(C_t, L_t) \qquad \text{s.t. for } t = 0, ..., \infty: \\
& 1 \geq L_t + N_t \\
& Y_t \geq C_t + I_t \\
& K_{t+1} = I_t + (1-\delta)K_t \\
& Y_t = F(K_t, N_t) \\
& 0 \leq K_{t+1}, C_t \qquad 0 \leq L_t \leq 1 \qquad 0 \leq N_t \leq 1 \\
& \text{and given } K_0
\end{aligned}
$$

Note that the constraints have to hold in every period $t = 0, ..., \infty$ such that these are really sequences of constraints. The problem above is equivalent to (why?) the simpler constrained optimization problem,

$$
\max_{\{K_{t+1},N_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t u(F(K_t, N_t) + (1-\delta)K_t - K_{t+1}, 1 - N_t) \tag{3}
$$

$$
\text{s.t. } 0 \leq N_t \leq 1, 0 \leq K_{t+1} \text{ and given } K_0
$$

Keep in mind that the solution to the planner's problem as formulated above is a *sequence* $\{K_{t+1}, N_t\}_{t=0}^{\infty}$ that is determined in period $t = 0$. The first order conditions for an interior solution are the following equations, which have to hold for all $t > 0$:

$$
K_{t+1}: \quad -u_c(C_t, 1 - N_t) + \beta\left[u_c(C_{t+1}, 1 - N_{t+1})\left(F_k(K_{t+1}, N_{t+1}) + 1 - \delta\right)\right] = 0 \tag{4a}
$$

$$
N_t: \quad -u_l(C_t, 1 - N_t) + u_c(C_t, 1 - N_t)F_n(K_t, N_t) = 0 \tag{4b}
$$

where $C_t = F(K_t, N_t) + (1-\delta)K_t - K_{t+1}$. These equations, which are sometimes called *Euler equations*, implicitly describe a second order nonlinear difference equation in K for which there are many solutions for a given initial condition K_0. To find the unique optimum an additional boundary condition is required. Because of the infinite dimension of the optimization problem this boundary condition is the *transversality condition* given by

$$
\lim_{T\to\infty} \beta^T u_c(C_T, 1 - N_T)k_{T+1} = 0 \tag{4c}
$$

The transversality condition says that the discounted value of the limiting capital stock is zero. Conditions (4a) to (4c) are necessary conditions for an optimum. Because of our assumptions on u and F, the first order conditions and the transversality condition are also sufficient conditions for a maximum.[1] It turns out that the planner's problem for this simple environment has a *recursive* structure. Let $v(K_0)$ be the maximized value of the optimization problem in (3) for a given initial per capita capital stock K_0, i.e. it is the number that one obtains when substituting the optimal choice of $\{K_{t+1}, N_t\}_{t=0}^{\infty}$ into the lifetime utility function in (3). The recursive structure implies that for any current value of the *state variable* K_t, we can write:

$$v(K_t) = \max_{0 \leq K_{t+1}, 0 \leq N_t \leq 1} u(F(K_t, N_t) + (1-\delta)K_t - K_{t+1}, 1 - N_t) + \beta v(K_{t+1})$$

The solution to the recursive problem is given by the time-invariant function v that solves this functional equation, also called the Bellman equation. Since the problem is identical for every period t, one might just as well omit the time subscript and write

$$v(K) = \max_{0 \leq K', 0 \leq N \leq 1} u(F(K, N) + (1-\delta)K - K', 1 - N) + \beta v(K') \tag{5}$$

The value function $v(K)$ implies a set of *policy functions*, $K' = k(K)$ and $N = n(K)$, that map the current value of the *state*, in this case K, into values for the *controls*, in this case K' and N, in a way that achieves the value $v(K)$:

$$v(K) = u(F(K, n(K)) + (1-\delta)K - k(K), 1 - n(K)) + \beta v(k(K)) \tag{6}$$

The analysis of the value function and/or the policy functions is called *dynamic programming.*

The Euler equations can also be derived directly from the functional equation in (5). Suppose that the value function v is differentiable and suppose that the right hand side of (5) is always attained at an interior. The first order conditions for the maximum problem in (5) are

$$K': \quad -u_c(C, 1-N) + \beta v_k(K') = 0 \tag{7a}$$

$$N: \quad -u_l(C, 1-N) + u_c(C, 1-N) F_n(K, N) = 0 \tag{7b}$$

where $C = F(K, N) + (1-\delta)K - K'$. These equations describe a first order difference

[1]See Stokey, Lucas and Prescott (1989) Theorem 4.15 on page 98

equation to which there is a unique solution for a given K, so no boundary conditions are missing in this case. If we use the envelope condition for the maximum problem,

$$v_k(K) = u_c(C, 1-N)\,(F_k(K,N) + 1 - \delta) \tag{8}$$

Eliminating v_k between (8) and (7a) and letting $C = C_t$, $N = N_t$, and $K' = K_{t+1}$ reproduces the Euler equations in $(4a) - (4b)$.

In what follows, we will often describe a solution to an optimization problem as in (3) in terms of the time-invariant policy functions that solve the Euler conditions. In the case at hand, we would like to find the functions $K_{t+1} = k(K_t)$, $C_t = c(K_t)$, $N_t = n(K_t)$ such that the following conditions hold for every $t > 0$:

$$-u_c(c(K_t), 1-n(K_t)) + \beta\,[u_c(c(k(K_t)), 1-n(k(K_t)))\,(F_k(k(K_t), n(k(K_t))) + 1 - \delta)] = 0 \tag{9a}$$

$$-u_l(c(K_t), 1-n(K_t)) + u_c(c(K_t), 1-n(K_t))F_n(K_t, n(K_t)) = 0 \tag{9b}$$

where in addition $c(K_t) = F(K_t, n(K_t)) + (1-\delta)K_t - k(K_t)$. Starting from an initial condition K_0, we can iterate on these policy functions to generate sequences $\{K_{t+1}, N_t\}_{t=0}^{\infty}$ that solve the problem in (3). You should always keep in mind, however, that in many other models this Euler equations approach is not appropriate, whereas the dynamic programming approach in (5) is.[2]

1.3 Competitive Equilibrium

The allocation chosen by the social planner under (3) or (5) can in this case be interpreted as predictions about the behavior of market economies in competitive equilibrium. In other words, the decentralized equilibrium allocations are Pareto optimal. In general, the equivalence between the social planner's outcome and the decentralized equilibrium depends on whether the conditions underlying the fundamental welfare theorems are satisfied, e.g. no taxes, perfect competition and the absence of other distortions. Later we will see examples of models where the equivalence breaks down, and it is therefore useful to formulate the competitive equilibrium of the basic neoclassical model.

In the decentralized economy, each period households sell labor and capital services to firms and buy consumption goods produced by firms, consuming some and accumulating the rest as capital. Trades take place in competitive markets, which must clear in every period. The period t price of the final consumption good is p_t, the price of one hour of

[2] This is usually true for instance when a choice variable can take only discrete values, e.g. the labor decision might be constrained to working full time or not working at all.

labor services *in terms of the final consumption good* in period t is w_t (i.e. the *real wage*) and the price of renting one unit of capital *in terms of the final consumption good* is r_t (i.e. the *real rental rate*).

The Firms Consider one particular firm j out of a total of J firms where J is a fixed number. The value of firm j in period t is

$$Q_t^j = (q_t^j + \pi_t^j)S_t^j \tag{10}$$

where π_t^j is the dividend or profit per share and q_t^j is the share price. S_t^j is total number of shares issued by firm j. We can have $S_t^j = 1$ without loss of generality. The firm's profit in period t is

$$\pi_t^j = Y_t^j - r_t K_t^j - w_t N_t^j \tag{11}$$

Production of output Y_t^j is subject to the technological constraint

$$Y_t^j \leq F(K_t^j, N_t^j) \tag{12}$$

The *firm's problem* is to choose Y_t^j, K_t^j, N_t^j in every period t that maximizes the value of the firm in (10) subject to the technological constraint and appropriate nonnegativity constraints, taking as given the prices p_t and q_t^j. Note that there is no intertemporal dimension to the firm's problem and that it is equivalent to maximizing profits in (11).

The Households Consider one particular household i out of a total of I households where I is a large number. Each household has identical initial endowments K_0 and shares s_0^j in every firm j and identical preferences given by

$$U^i = \sum_{t=0}^{\infty} \beta^t u(C_t^i, 1 - N_t^i), \quad \beta < 1 \tag{13}$$

In each period t, the household faces the budget constraint

$$C_t^i + K_{t+1}^i - (1-\delta)K_t^i + \sum_J q_t^j s_{t+1}^{ij} \leq w_t N_t^i + r_t K_t^i + \sum_J \left(q_t^j + \pi_t^j\right) s_t^{ij} \tag{14}$$

in addition to all the appropriate nonnegativity constraints.

The *household's problem* is to choose sequences of $\{C_t^i, K_{t+1}^i, N_t^i, s_{t+1}^{i1}, ..., s_{t+1}^{iJ}\}_{t=0}^{\infty}$ subject to the budget constraint and nonnegativity constraints, taking as given the prices

$\{p_t, r_t, w_t, q_t^1, ..., q_t^J\}_{t=0}^{\infty}$, dividend streams $\{\pi_t^1, ..., \pi_t^J\}_{t=0}^{\infty}$ and initial endowments K_0, $s_0^1, ..., s_0^J$.

Competitive Equilibrium A *competitive equilibrium* is described by the allocations $\{C_t^i, K_{t+1}^i, N_t^i, s_{t+1}^{i1}, ..., s_{t+1}^{iJ}\}_{t=0}^{\infty}$ for $i = 1, ..., I$, $\{Y_t^j, K_t^j, N_t^j\}_{t=0}^{\infty}$ for $j = 1, ..., J$ and prices $\{p_t, r_t, w_t, q_t^1, ..., q_t^J\}_{t=0}^{\infty}$ such that

- the allocations solve the households' problem for all i
- the allocations solve the firms' problem for all j
- in every period t all markets clear, i.e.

$$\sum_J Y_t^j = \sum_I \left(C_t^i + K_{t+1}^i - (1-\delta)K_t^i\right) \tag{15a}$$

$$\sum_I N_t^i = \sum_J N_t^j \tag{15b}$$

$$\sum_I K_t^i = \sum_J K_t^j \tag{15c}$$

$$\sum_I s_t^{ij} = 1, \quad j = 1, ..., J \tag{15d}$$

EXERCISE: Show that 1) sequences that solve (4a)-(4c) are competitive equilibrium allocations; 2) the value of the firm equals the present discounted value of future dividend streams. Consider the alternative assumption that the firms own the capital stock and repeat 1) and 2).

1.4 Approximate Linear Dynamics in the Neighborhood of the Steady State

In this section, we turn to the problem of finding the functions $K_{t+1} = k(K_t)$, $C_t = c(K_t)$, $N_t = n(K_t)$ such that the following conditions hold for every $t > 0$:

$$-u_c(C_t, 1-N_t) + \beta\left[u_c(C_{t+1}, 1-N_{t+1})\left(F_k(K_{t+1}, N_{t+1}) + 1 - \delta\right)\right] = 0 \tag{16a}$$

$$-u_l(C_t, 1-N_t) + u_c(C_t, 1-N_t)F_n(K_t, N_t) = 0 \tag{16b}$$

$$C_t + K_{t+1} - (1-\delta)K_t - F(K_t, N_t) = 0 \tag{16c}$$

In general, there are no analytical solutions and we have to resort to numerical approximation of the policy functions. There are many different approximation methods, but here we will focus on simple (log-)linearization methods.[3] In cases where the competitive equilibrium (CE) is not Pareto optimal (i.e., when there are frictions), loglinear approximations of the CE conditions are often easy because they are calculated using the optimality conditions of the problem, whereas other methods are usually harder to implement. That is why loglinear approximations are widely used in practice.

The basic trick is to approximate the system of nonlinear difference equations by a system of linear difference equations using first order Taylor expansions around a certain point. Loglinearization is therefore a *local* approximation method and in dynamic models, the approximation point is almost always the (deterministic) steady state. There are two caveats to using local approximation techniques that are worth mentioning: 1) some models do not possess a steady state and others have multiple steady states, 2) local methods are inappropriate when problems involve large perturbations away from the approximation point and dynamic paths are significantly nonlinear.

The Steady State The steady state level of capital $\bar{K}$, labor input $\bar{N}$ and consumption $\bar{C}$ is implicitly given by

$$\begin{aligned} F_k(\bar{K},\bar{N}) &= \frac{1}{\beta} - 1 + \delta \\ F_n(\bar{K},\bar{N}) &= \frac{u_l(\bar{C},1-\bar{N})}{u_c(\bar{C},1-\bar{N})} \\ \bar{C} + \delta\bar{K} &= F(\bar{K},\bar{N}) \end{aligned}$$

The Linear System of Difference Equations Define $\hat{x}_t = \log\left(\frac{X_t}{\bar{X}}\right)$. In the neighborhood of the steady state value $\bar{X}$, $\hat{x}_t$ has the interpretation of the percentage deviation from $\bar{X}$. We linearize each condition in (16a)-(16c) in terms of deviations from the steady state:

$$\xi_{cc}\hat{c}_t - \xi_{cl}\frac{\bar{N}}{1-\bar{N}}\hat{n}_t = \xi_{cc}\hat{c}_{t+1} - \xi_{cl}\frac{\bar{N}}{1-\bar{N}}\hat{n}_{t+1} + (1-\beta(1-\delta))\left(\eta_{kn}\hat{n}_{t+1} + \eta_{kk}\hat{k}_{t+1}\right) \tag{17a}$$

$$\eta_{nn}\hat{n}_t + \eta_{nk}\hat{k}_t = (\xi_{lc} - \xi_{cc})\,\hat{c}_t + (\xi_{cl} - \xi_{ll})\,\frac{\bar{N}}{1-\bar{N}}\hat{n}_t \tag{17b}$$

$$s_c\hat{c}_t + \frac{s_i}{\delta}\hat{k}_{t+1} = \left((1-\alpha) + s_i\frac{1-\delta}{\delta}\right)\hat{k}_t + \alpha\hat{n}_t \tag{17c}$$

where ξ_{ab} the elasticity of marginal utility of a with respect to b evaluated at the steady state; η_{ab} is the elasticity of the marginal product of a with respect to b, evaluated at the

[3]See Stokey, Lucas and Prescott (1989) section 6.3 and 6.4. See also Canova (2007), section 2.2 for discussion of various approximation methods.

steady state; s_c and s_i are the consumption and investment shares in output in the steady state and α is the output elasticity of labor input.

Substituting out $\hat{n}_t$, we can write a first order dynamic system in $\hat{c}_{t+1}$ and $\hat{k}_{t+1}$:

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} = W \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} \tag{18}$$

To compute the solution to this difference equation and to examine its properties, we use the decomposition $W = P\lambda P^{-1}$ where P is the matrix of eigenvectors and λ is a diagonal matrix with eigenvalues on the diagonal. The general solution to the difference equation for initial conditions $\hat{c}_0$ and $\hat{k}_0$ is given by

$$\begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} = W^t \begin{bmatrix} \hat{k}_0 \\ \hat{c}_0 \end{bmatrix} = P\lambda^t P^{-1} \begin{bmatrix} \hat{k}_0 \\ \hat{c}_0 \end{bmatrix} \tag{19}$$

Solving for the eigenvalues, you will notice that one of them is smaller then one and the other exceeds 1. Therefore it follows that the system is on an explosive path for arbitrary $\hat{c}_0$, violating the boundary conditions. There is a specific value of $\hat{c}_0$ that results in (19) satisfying the boundary conditions. This particular solution specifies the unique optimal time paths for $\{\hat{c}_t\}_{t=0}^{\infty}$ and $\{\hat{k}_{t+1}\}_{t=0}^{\infty}$ from which the optimal sequence for output, labor, etc. can be computed.

A convenient way to get the linear policy functions is *the method of undetermined coefficients*: Let's guess solutions: $\hat{k}_{t+1} = \phi_1 \hat{k}_t$ and $\hat{c}_t = \phi_2 \hat{k}_t$ and substitute these in (18) to obtain

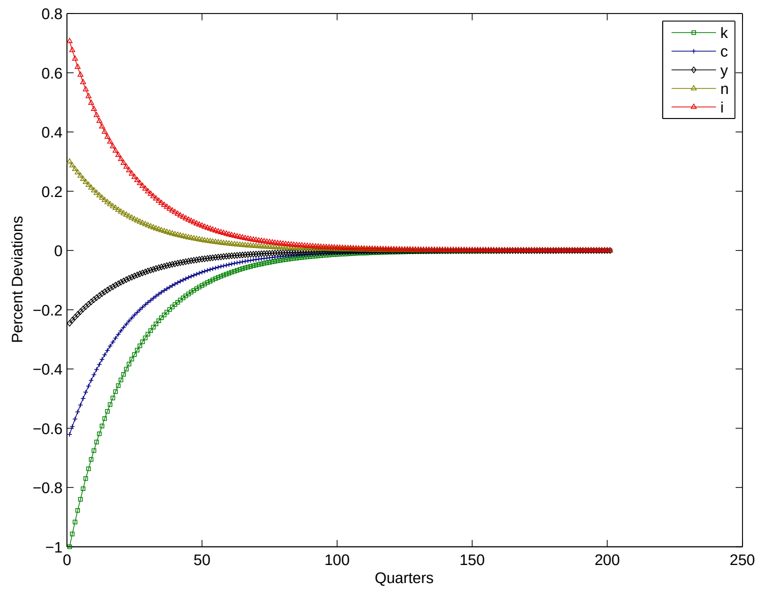
$$\begin{bmatrix} \phi_1 \\ \phi_2\phi_1 \end{bmatrix} \hat{k}_t = \begin{bmatrix} w_{11} + w_{12}\phi_2 \\ w_{21} + w_{22}\phi_2 \end{bmatrix} \hat{k}_t \tag{20}$$

Therefore, the coefficients ϕ_1 and ϕ_2 are implicitly given by

$$\begin{aligned} \phi_1 &= w_{11} + w_{12}\phi_2 \\ \phi_1\phi_2 &= w_{21} + w_{22}\phi_2 \end{aligned}$$

ECERCISE: Solve for the eigenvalues of W. Show that ϕ_1 equals the smallest (in absolute value) eigenvalue of W and find ϕ_2. Find the policy functions for $\hat{n}_t$, $\hat{i}_t$ and $\hat{y}_t$.

Figure 1: Transition dynamics, percentage deviations



Transition dynamics: A numerical example (growthmodel.m) Assume the following functional forms:

$$u(C_t, L_t) = \log(C_t) + \theta_l \log(L_t) \tag{22a}$$

$$Y_t = AK_t^{1-\alpha} N_t^{\alpha} \tag{22b}$$

with parameter values are $\alpha = 0.58$, $\beta = 0.988$, $\delta = 0.025$. The parameter θ_l is chosen to obtain a steady state value for N of 0.20. The parameter A is chosen to normalize the steady state value of output Y to 1. The linearized system is

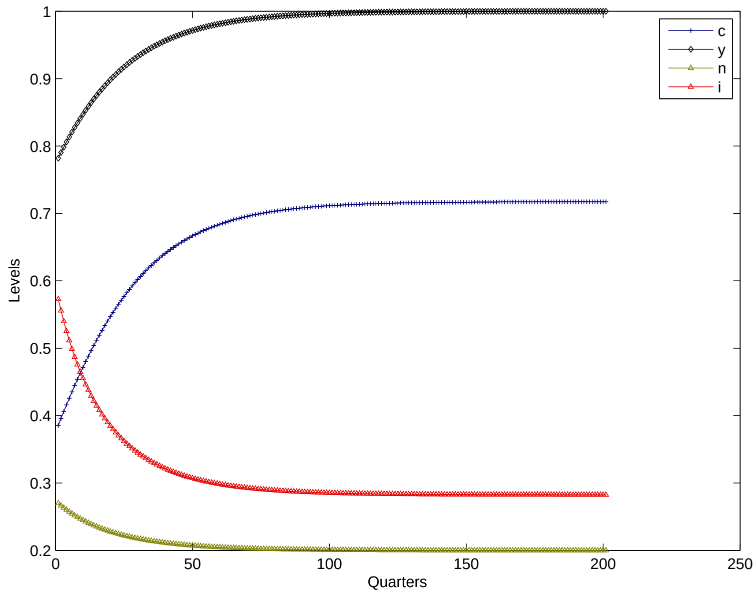
$$-\hat{c}_t = -\hat{c}_{t+1} + (1 - \beta(1-\delta)) \left(\alpha \hat{n}_{t+1} - \alpha \hat{k}_{t+1}\right) \tag{23a}$$

$$-(1-\alpha)\hat{n}_t + (1-\alpha)\hat{k}_t = \hat{c}_t + \frac{\bar{N}}{1-\bar{N}}\hat{n}_t \tag{23b}$$

$$s_c \hat{c}_t + \frac{s_i}{\delta}\hat{k}_{t+1} = \left(1 - \alpha + s_i \frac{1-\delta}{\delta}\right)\hat{k}_t + \alpha \hat{n}_t \tag{23c}$$

Figure 1 and 2 plot the transition paths in percentage deviations and in levels, for a situation where the initial capital stock is 1 percent below its steady state level.

Figure 2: Transition dynamics, levels



1.5 Balanced Growth Path

A characteristic of most industrialized countries is that output per capita, consumption per capita, investment per capita and other variables exhibit growth over long periods of time. This long-run growth occurs at rates that are roughly constant over time within economies but differ across economies. This pattern suggest *steady state growth*, which means that the levels of certain key variables grow at a constant rate. In that case, we say there is a *balanced growth path*. It is possible to introduce this feature into the model above, but only under certain restrictions on preferences and technologies will the model dynamics be consistent with balanced growth.

Restrictions on production For a steady state to be feasible, technical change must be expressible in a labor augmenting form. Consider the following Cobb-Douglas production function

$$F(K_t, N_t) = AK_t^{1-\alpha}(X_t N_t)^{\alpha} \tag{24}$$

where $0 < \alpha < 1$. *Permanent* technological variations are restricted to be in labor productivity X_t. The quantity $X_t N_t$ is usually referred to as *effective labor units*. Denote one plus the growth rate of a variable Z_t by γ_z, i.e.

$$\frac{Z_{t+1}}{Z_t} = \gamma_z$$

Since the labor endowment is bounded by the endowment, γ_n must be one, i.e. time devoted to working does not exhibit long run growth. The capital accumulation equation (2) implies that $\gamma_i = \gamma_k$. The production function (24) and the resource constraint then imply that in any feasible steady state it is required that $\gamma_c = \gamma_i = \gamma_y = \gamma_x$. To summarize, in a technologically feasible steady state per capita capital, investment, consumption and output must grow at the rate of labor augmenting technological progress γ_x, whereas hours worked is constant. Note that in any such feasible steady state, the marginal product of capital and the marginal product of a unit of labor input in efficiency units are constant.

Restrictions on preferences The technologically feasible steady state growth rates must also be consistent with the households' optimality conditions. Two restrictions are required:

1. the intertemporal elasticity of substitution in consumption must be invariant to the scale of consumption
2. the income and substitution effects associated with sustained growth in labor productivity must not alter labor supply

These conditions imply the following class of admissible utility functions

$$u(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} v(L_t)$$

for $0 < \sigma < 1$ and $\sigma > 1$ and

$$u(C_t, L_t) = \log(C_t) + v(L_t)$$

for $\sigma = 1$. Of course there are additional restrictions on the function v to ensure that consumption and leisure are goods and that utility is concave.[4] The constant intertemporal elasticity of substitution in consumption is $1/\sigma$.

The standard method of analyzing models with steady state growth is to transform the economy into a stationary one where the dynamics are more amenable to analysis. A *stationarity inducing transformation* is dividing all variables in the system by the growth

[4]See King, Plosser and Rebelo (1988) footnote 11.

component X_t, such that $c_t = C_t/X_t$, $i_t = I_t/X_t$, $y_t = Y_t/X_t$ and $k_{t+1} = K_{t+1}/X_{t+1}$. The Euler conditions in the transformed economy are:

$$-u_c(c_t, 1 - N_t) + \beta\gamma_x^{-\sigma}\left[u_c(c_{t+1}, 1 - N_{t+1})\left((1-\alpha)A\left(\frac{k_{t+1}}{N_{t+1}}\right)^{-\alpha} + 1 - \delta\right)\right] = 0 \quad (25a)$$

$$-u_l(c_t, 1 - N_t) + u_c(c_t, 1 - N_t)\alpha A\left(\frac{k_t}{N_t}\right)^{1-\alpha} = 0 \quad (25b)$$

where $c_t = Ak_t^{1-\alpha}N_t^{\alpha} + (1-\delta)k_t - \gamma_x k_{t+1}$ and where $\beta\gamma_x^{1-\sigma} < 1$ is required throughout to guarantee finiteness of lifetime utility. Note that this economy is equivalent to the no-growth economy described earlier with two exceptions: first, the parameter γ_x enters the capital accumulation equation and second, the effective rate of time preference is altered.

Transition dynamics with growth (growthmodel.m) Consider the same numerical example as before and let $\gamma_x = 1.004$ and $\beta = 0.988$. The linearized system is

$$-\hat{c}_t = -\hat{c}_{t+1} + \left(1 - \frac{\beta}{\gamma_x}(1-\delta)\right)\left(\alpha\hat{n}_{t+1} - \alpha\hat{k}_{t+1}\right) \quad (26a)$$

$$-(1-\alpha)\hat{n}_t + (1-\alpha)\hat{k}_t = \hat{c}_t + \frac{\bar{N}}{1-\bar{N}}\hat{n}_t \quad (26b)$$

$$s_c\hat{c}_t + s_i\frac{\gamma_x}{\delta + \gamma_x - 1}\hat{k}_{t+1} = \left(1 - \alpha + s_i\frac{1-\delta}{\delta + \gamma_x - 1}\right)\hat{k}_t + \alpha\hat{n}_t \quad (26c)$$

Figure 3 and 4 plot the transition paths in percentage deviations and in levels, for a situation where the initial capital stock is 1 percent below its steady state level.

Figure 3: Transition dynamics in model with steady state growth, percentage deviations

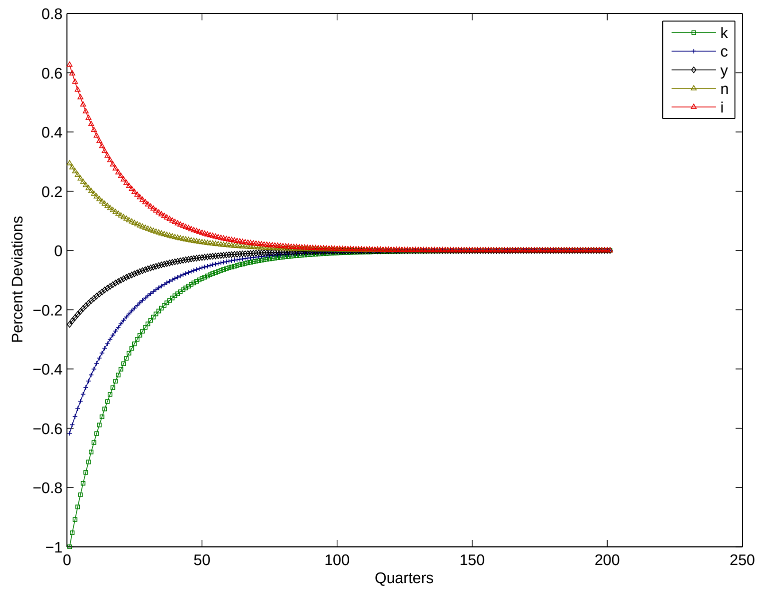
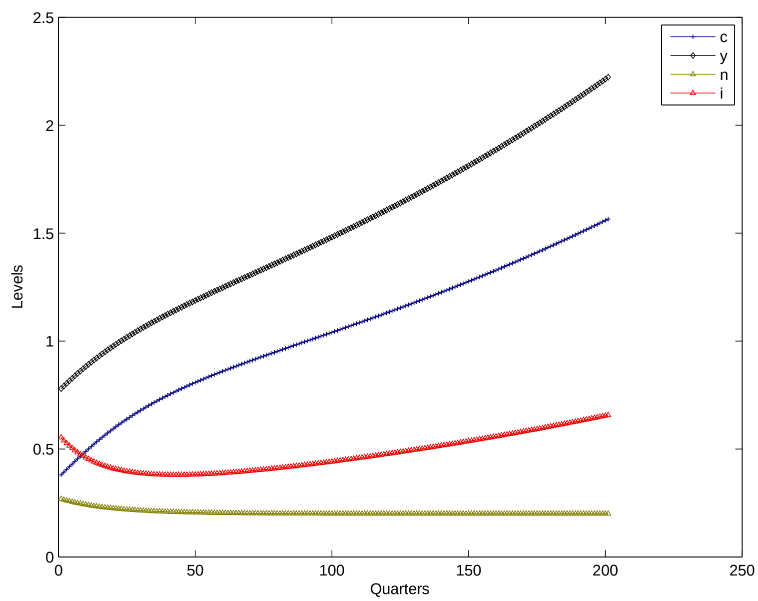


Figure 4: Transition dynamics in model with steady state growth, levels



2 Introducing Uncertainty: The Real Business Cycle Model

The dynamics implied by the neoclassical growth model still do not resemble the time series behavior of the main macroeconomic variables at business cycle frequencies. The next step is to introduce stochastic shocks into the neoclassical framework that are thought to drive business cycle fluctuations. Extending the neoclassical growth model by including exogenous stochastic shocks leads to the *real business cycle (RBC) model*, which belongs to the class of *dynamic stochastic general equilibrium (DSGE)* models. The shocks that can be included range from exogenous changes in technology, tax rates and government spending, tastes, government regulation, terms of trade to energy prices or others. The 'real' in RBC refers to the fact that the shocks are nonmonetary in nature.[5] In this section, we will restrict attention to transitory shocks to technology.

2.1 The RBC Model

Consider the production function

$$F(K_t, N_t) = A_t K_t^{1-\alpha}(X_t N_t)^{\alpha}$$

where the only difference with the model considered before is the time subscript on A. We therefore now permit temporary changes in total factor productivity through A_t, which evolves according to an exogenous stationary stochastic process. The typical specification used for A_t is

$$\begin{aligned} A_t &= \bar{A}e^{a_t} \\ a_t &= \rho a_{t-1} + \epsilon_t \end{aligned} \tag{27}$$

where ϵ_t is a white noise random variable with variance σ_ϵ^2 and $0 < \rho < 1$ measures the shock persistence. The planner's maximization problem can now be written as follows

$$\max_{\{K_{t+1}, N_t\}_{t=0}^{\infty}} \quad E\left[\sum_{t=0}^{\infty} \beta^t u(F(K_t, N_t) + (1-\delta)K_t - K_{t+1}, 1 - N_t) \mid I_0\right] \tag{28}$$

$$\text{s.t. } 0 \leq N_t \leq 1, 0 \leq K_{t+1} \text{ and given } K_0, A_0$$

where I_0 denotes the information set available at time 0. The assumption is that the current and lagged realizations of a_t belong to the agent's information set at period t, but

[5] We will turn to monetary shocks later in this course.

future realizations do not. Let the *history* of events up to time t be denoted by $a^t = [a_t, a_{t-1}, ..., a_0]$. In general, the solution will now consist of a set of sequences $\{K_{t+1}, N_t\}_{t=0}^{\infty}$ for every possible realization of histories $\{a^t\}_{t=0}^{\infty}$. Because of the law of motion for a_t assumed in (27), a_t is a first-order Markov process. Therefore, no information other than a_t is useful in making predictions and the solution can be characterized by plans for K_{t+1} and C_t that are contingent on the state variables K_t and a_t only and not on the entire history of shocks.[6] A recursive formulation of the planning problem is

$$v(K,a) = \max_{0 \leq K', 0 \leq N \leq 1} u(F(K,N) + (1-\delta)K - K', 1-N) + \beta E\left[v(K',a') \mid a\right] \tag{29}$$

where the expectation is conditional on a.

The transversality and first-order conditions for the sequence problem are

$$-u_c(C_t, 1-N_t) + \beta E_t \left[u_c(C_{t+1}, 1-N_{t+1}) \left((1-\alpha) A_{t+1} \left(\frac{K_{t+1}}{X_{t+1}N_{t+1}} \right)^{-\alpha} + 1 - \delta \right) \right] = 0 \tag{30a}$$

$$-u_l(C_t, 1-N_t) + u_c(C_t, 1-N_t)\alpha A_t \left(\frac{K_t}{X_t N_t} \right)^{1-\alpha} X_t = 0 \tag{30b}$$

$$\lim_{T \to \infty} \beta^T E_t \left[u_c(C_T, 1-N_T) K_{T+1} \right] = 0 \tag{30c}$$

where $C_t = A_t K_t^{1-\alpha}(X_t N_t)^{\alpha} + (1-\delta)K_t - K_{t+1}$. E_t is the expectation operator conditional on the information available at time t. After conducting the same stationarity inducing transformation as before, we formulate the solution to the RBC model as a set of time invariant functions $k_{t+1} = k(k_t, a_t)$, $c_t = c(k_t, a_t)$ and $N_t = n(k_t, a_t)$ that solve the system of stochastic difference equations:

$$-u_c(c_t, 1-N_t) + \beta \gamma_x^{-\sigma} E_t \left[u_c(c_{t+1}, 1-N_{t+1}) \left((1-\alpha) A_{t+1} \left(\frac{k_{t+1}}{N_{t+1}} \right)^{-\alpha} + 1 - \delta \right) \right] = 0 \tag{31a}$$

$$-u_l(c_t, 1-N_t) + u_c(c_t, 1-N_t)\alpha A_t \left(\frac{k_t}{N_t} \right)^{1-\alpha} = 0 \tag{31b}$$

$$c_t + \gamma_x k_{t+1} - (1-\delta)k_t - A_t k_t^{1-\alpha}(N_t)^{\alpha} = 0 \tag{31c}$$

As before, only in exceptional cases do analytical solutions exist and in general it is necessary to resort to numerical approximations of the solution. We will continue to work with loglinear approximation methods. First, we must settle on a point around which to conduct the first-order Taylor expansion. It is important to realize that in models with stochastic shocks, the steady state is characterized by stationary distributions and not by points.

[6] See Ljunqvist and Sargent (2004), chapter 12, for a formulation of the general case.

Common practice is nevertheless to choose the steady state of a deterministic version of the model as the approximation point, which in our case is the steady state of the neoclassical growth model. You should be aware that it is never assured that a deterministic steady state is indeed an interesting place to analyze dynamics, as it might have a low unconditional probability in the stochastic model. Examples include problems with potentially binding inequality constraints (e.g., borrowing or irreversibility constraints), since the nonstochastic steady state ignores these. Besides the caveats of using local methods mentioned above, another important assumption of the loglinear approximation method is that it imposes *certainty equivalence*. This principle allows us to eliminate the expectation operator in (31a) and reinsert it in front of all future unknown variables once a solution is found. This operation is possible because the covariance matrix of the stochastic shocks does not enter the linear policy rules. As a result, first-order approximations are insufficient when evaluating welfare across policies that do not affect the steady state, when analyzing asset pricing problems, or when risk considerations become important.[7]

The linearized optimality conditions are

$$\xi_{cc}\hat{c}_t - \xi_{cl}\frac{\bar{N}}{1-\bar{N}}\hat{n}_t = E_t\left[\xi_{cc}\hat{c}_{t+1} - \xi_{cl}\frac{\bar{N}}{1-\bar{N}}\hat{n}_{t+1} + \left(1-\beta\gamma_x^{-\sigma}(1-\delta)\right)\left(a_{t+1}+\alpha\hat{n}_{t+1}-\alpha\hat{k}_{t+1}\right)\right] \tag{32a}$$

$$a_t - (1-\alpha)\hat{n}_t + (1-\alpha)\hat{k}_t = (\xi_{lc}-\xi_{cc})\,\hat{c}_t + (\xi_{cl}-\xi_{ll})\,\frac{\bar{N}}{1-\bar{N}}\hat{n}_t \tag{32b}$$

$$s_c\hat{c}_t + s_i\frac{\gamma_x}{\delta+\gamma_x-1}\hat{k}_{t+1} = a_t + \left(1-\alpha+s_i\frac{1-\delta}{\delta+\gamma_x-1}\right)\hat{k}_t + \alpha\hat{n}_t \tag{32c}$$

Substituting out $\hat{n}_t$ and using certainty equivalence, we can write the dynamic system as

$$E_t\begin{bmatrix}\hat{k}_{t+1}\\ a_{t+1}\\ \hat{c}_{t+1}\end{bmatrix} = W\begin{bmatrix}\hat{k}_t\\ a_t\\ \hat{c}_t\end{bmatrix}$$

$$W = \begin{bmatrix} w_{11} & w_{12} & w_{13}\\ 0 & \rho & 0\\ w_{31} & w_{32} & w_{33}\end{bmatrix}$$

[7] See Canova (2007) section 2.2 for further discussion of these issues.

Let's guess solutions: $\hat{k}_{t+1} = \phi_{11}\hat{k}_t + \phi_{12}a_t$ and $\hat{c}_t = \phi_{21}\hat{k}_t + \phi_{22}a_t$ such that

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ 0 & \rho \\ \phi_{21}\phi_{11} & \phi_{21}\phi_{12} + \phi_{22}\rho \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ a_t \end{bmatrix} = \begin{bmatrix} w_{11} + w_{13}\phi_{21} & w_{12} + w_{13}\phi_{22} \\ 0 & \rho \\ w_{31} + w_{33}\phi_{21} & w_{32} + w_{33}\phi_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ a_t \end{bmatrix}$$

and solve for the ϕ's from

$$\begin{aligned} \phi_{11} &= w_{11} + w_{13}\phi_{21} \\ \phi_{12} &= w_{12} + w_{13}\phi_{22} \\ \phi_{21}\phi_{11} &= w_{31} + w_{33}\phi_{21} \\ \phi_{21}\phi_{12} + \phi_{22}\rho &= w_{32} + w_{33}\phi_{22} \end{aligned}$$

As before, the solution to this system is not unique and we select the one that satisfies the transversality condition.

After solving for the ϕ's it is straightforward to obtain the policy functions for investment, output, hours and any other variables of interest. It is useful to write the solution in the following form

$$\begin{aligned} s_{t+1} &= Gs_t + Fe_{t+1} \\ z_t &= Hs_t \end{aligned} \tag{33}$$

where s_t is a $m \times 1$ vector of state variables, e_{t+1} is an $l \times 1$ vector of exogenous disturbances, z_t is an $n \times 1$ vector of variables of interest, G is $m \times m$, F is $m \times l$ and H is $n \times m$. m is the number of state variables, l is the number of exogenous shocks and n is the number of variables of interest. In our simple RBC model, we have $m = 2$ and $s_t = [\hat{k}_t, a_t]'$, $l = 1$ and $e_{t+1} = \epsilon_{t+1}$, and for instance $n = 4$ with $z_t = [\hat{y}_t, \hat{c}_t, \hat{i}_t, \hat{n}]'$.

Impulse Responses Impulse response functions provide information on the system's average conditional response to a technology shock at date t. The response of the system in period $t + k$ to an impulse in the j-th element of e_{t+1} at date t+1 is

$$\begin{aligned} s_{t+k} - E\left[s_{t+k} \mid s_t\right] &= G^{k-1}F\epsilon^j_{t+1} \\ z_{t+k} - E\left[z_{t+k} \mid s_t\right] &= HG^{k-1}F\epsilon^j_{t+1} \end{aligned} \tag{34}$$

where ϵ^j_{t+1} is a $l \times 1$ vector with the j-th element of e_{t+1} as the only non-zero element.

Model Simulation Model simulation is done by iterating on (33) for a given random sequence of shocks e.

A numerical example (rbcmodel.m) Consider again the utility function

$$u(C_t, L_t) = \log(C_t) + \theta_l \log(L_t)$$

As before, the preference and technology parameters are $\alpha = 0.58$, $\beta = 0.988$, $\delta = 0.25$, $\gamma_x = 1.004$ and $\beta = 0.988$, θ_l is chosen to obtain a steady state value for N of 0.20 and A is chosen to normalize the steady state value of output Y to 1. The autoregressive parameter of the technology shock process is $\rho = 0.9$. This parametrization corresponds to the "Long-Plosser with realistic depreciation" case in King et al. (1988). The linearized system is

$$-\hat{c}_t = E_t\left[-\hat{c}_{t+1} + \left(1 - \frac{\beta}{\gamma_x}(1-\delta)\right)\left(a_{t+1} + \alpha\hat{n}_{t+1} - \alpha\hat{k}_{t+1}\right)\right] \tag{35a}$$

$$a_t - (1-\alpha)\hat{n}_t + (1-\alpha)\hat{k}_t = \hat{c}_t + \frac{\bar{N}}{1-\bar{N}}\hat{n}_t \tag{35b}$$

$$s_c\hat{c}_t + s_i\frac{\gamma_x}{\delta + \gamma_x - 1}\hat{k}_{t+1} = a_t + \left(1 - \alpha + s_i\frac{1-\delta}{\delta+\gamma_x-1}\right)\hat{k}_t + \alpha\hat{n}_t \tag{35c}$$

Figure 5 displays the impulse responses to a 1% positive shock to technology and figure 6 gives an example of simulated time series produced by the RBC model, where the standard deviation of the technology innovation is $\sigma_\epsilon = 0.01$.

Figure 5: Impulse Response to a 1% positive shock to technology

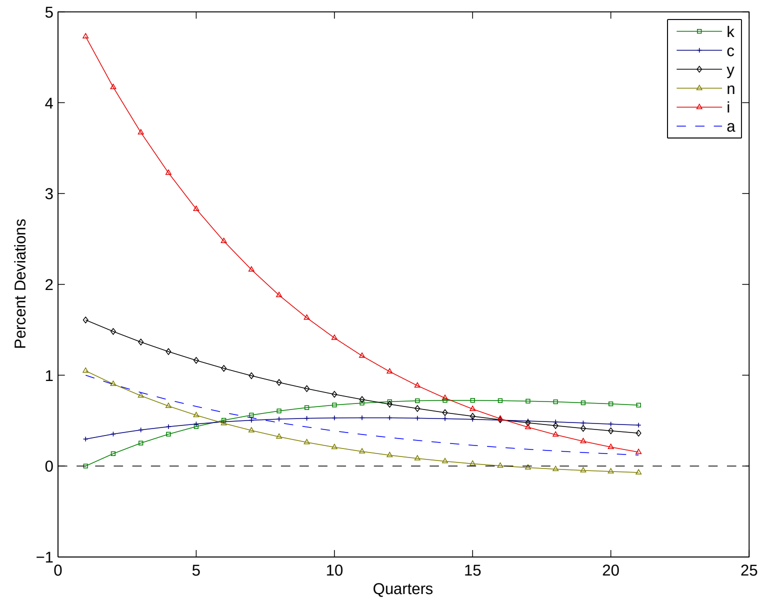
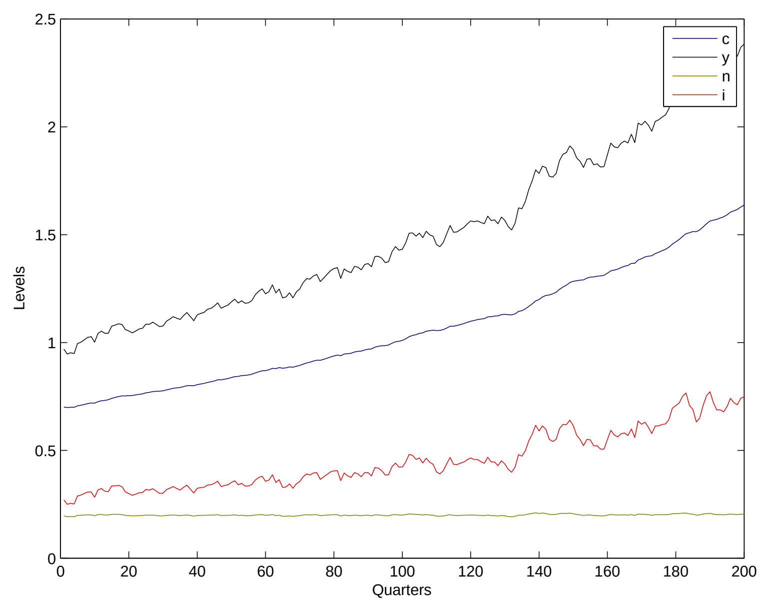


Figure 6: Simulated Series



2.2 Growth Accounting

Given our Cobb-Douglas production function

$$Y_t = A_t K_t^{1-\alpha}(X_t N_t)^\alpha$$

we can write

$$\log Y_t = \log A_t + \alpha \log X_t + (1-\alpha)\log(K_t) + \alpha \log(N_t)$$

The *Solow residual* is defined as

$$\begin{aligned} \log SR_t &= \log Y_t - (1-\alpha)\log(K_t) - \alpha \log(N_t) \\ &= \log A_t + \alpha \log X_t \end{aligned}$$

Note that the Solow residual can be measured empirically using time series on output, capital and hours. A series for the capital stock can be constructed using the capital accumulation law for a given choice of δ and k_0 and data on investment. In the simple RBC model, the Solow residual therefore gives us a measure of the technology process.

2.3 Matching Theory and Data: Calibration and Model Analysis

Calibration in General *Calibration*, as defined by Canova (2007), is a collection of procedures designed to provide an answer to economic questions by using a model that approximates the data generating process (DGP) of (a subset of) the observable data. The essence of the methodology can be summarized as follows:

1. Choose an economic question to be addressed. Typically they are of the form:

   a) How much of fact X can be explained with impulses of type Y?

   b) Is it possible to generate features F by using theory T?

   c) Can we reduce the discrepancy D of the theory from the data by using feature F?

   d) How much do endogenous variables change if the process for the exogenous variables is altered?

2. Select a model design which bears some relevance to the question asked.

3. Choose functional forms for the primitives of the model and find a solution for the endogenous variables in terms of the exogenous ones and the parameters.

4. Evaluate the quality of the model by comparing its outcomes with a set of "stylized facts" of the actual data.

5. Propose an answer to the question, characterize the uncertainty surrounding the answer, and do policy analysis if required.

Stylized facts can be sample statistics (standard deviations, correlations), histograms, VAR coefficients, likelihood function, structural impulse responses, etc. In a strict sense, all models are approximations to the true DGP and are in that way false and unrealistic. A calibrator is satisfied with her effort if, through a process of theoretical respecification, the model captures an increasing number of features or stylized facts of the data while maintaining a highly stylized structure (Canova (2007)). This model can be realistically used as a laboratory to conduct experiments. For instance, a model that has matched reasonably well what happened in previous tax reforms could be a reliable instrument to ask what would happen in a new tax reform.

The controversial part is the selection of parameter values and stochastic processes of the model. Here is the common approach in DSGE modeling:

1. Choose parameters such that the deterministic steady state for the endogenous variables replicates the time series averages of the actual economy.

2. Some parameters cannot be pinned down by the deterministic steady state. There are different strategies to choose the remaining parameters:

   a) use available, often microeconometric, estimates of these parameters

   b) obtain informal estimates using a method of moments

   c) obtain formal estimates using *Generalized Method of Moments* (GMM), *Maximum Likelihood* (ML) or *Simulated Method of Moments* (SMM).

If the selection process leaves some of the parameters undetermined, it is often useful to conduct a *sensitivity analysis* to determine how the outcomes vary when these parameters are changed.

Calibrating our RBC Model Let's now conduct a calibration exercise and conduct a model evaluation of our RBC model along the lines of King et al. (1988). The time period in the model is chosen to correspond to one quarter, the frequency of our US dataset. We will adopt the utility specification

$$u(C_t, L_t) = \log(C_t) + \frac{\theta_l}{1-\xi} L_t^{1-\xi}$$

which implies zero cross elasticities $\xi_{lc} = \xi_{cl} = 0$ and unitary elasticity in consumption $\sigma = -\xi_{cc} = 1$.

The parameters $\alpha, \beta, \gamma_x, \delta$ and θ_l can be calibrated such that the deterministic steady state replicates certain time series averages of actual US data.

The labor share α In competitive equilibrium, the real wage rate and the equals the marginal product of labor, i.e. $w_t = \alpha \frac{Y_t}{N_t}$. This implies that total wage income is $w_t N_t = \alpha Y_t$. Similarly, capital income is $(1-\alpha)Y_t$. In other words, α measures the share of labor income in GDP. Under the Cobb-Douglas assumption, this share is a constant, which is usually motivated by the stylized fact in growth theory that labor shares are invariant to the scale of economic activity. However, with labor augmenting technological progress, this is true for any constant returns to scale production function. The parameter α can be chosen to match the average ratio of wage income to real GDP in US data, yielding a value of approximately $\alpha = 0.58$.

The technological growth rate $\gamma_x - 1$ The technological growth rate can be obtained from the average common trend growth rate of output, consumption and investment, of about 1.6% annually. This implies $\gamma_x = (1 + 0.016)^{0.25} = 1.004$.

The discount factor β The average real return to equity, which in the model corresponds to $r + 1 - \delta$, is about 6.5% per annum in the US. Note that the effective discount factor in the model with growth is $\beta\gamma_x^{1-\sigma} = \gamma_x/(r+1-\delta) = 1.016^{0.25}/1.065^{0.25} \approx 0.988$, from which the value for β follows.

The depreciation rate δ First, note that the average level of TFP $\bar{A}$ does not play any role in the model dynamics and is just a scaling factor. Hence, without loss of generality we can normalize output in the deterministic steady state to unity. We can look up the average ratio of investment to GDP in the US, which we will take to be 0.295. From the capital accumulation equation and the normalization of output to unity, we have that this I/Y ratio is given by

$$\bar{I} = (\gamma_x + \delta - 1)\bar{K}$$

From the investment Euler equation we have that

$$1 = \frac{\beta}{\gamma_x}\left[\frac{1-\alpha}{\bar{K}} + 1 - \delta\right]$$

Combining these we can solve for $\delta = 0.025$, which corresponds to an annual depreciation rate of 10%.

The elasticity of the marginal utility of leisure with respect to leisure $\xi_{ll} = -\xi$, as well as the shock persistence ρ and variance σ_ϵ are uninformative for the deterministic steady state and must therefore be chosen in another way.

The leisure preference parameters ξ and θ_l First, we pin down $\bar{N} = 0.20$, which is the fraction of the time endowment spent in the workplace in the deterministic steady state. This value corresponds to the average workweek as a fraction of total weekly hours in US data. For any given value of ξ, the leisure parameter in the utility function follows from the labor supply condition

$$\theta_l = \alpha\frac{(1-\bar{N})^\xi}{\bar{N}}\frac{1}{1-\bar{I}}$$

which leaves us with the question what value to choose for ξ. Note that ξ is closely related the wage elasticity of labor supply, which equals $\frac{1-\bar{N}}{\bar{N}}/\xi$. Microeconometric estimates suggest that the wage elasticity of labor supply is very low, at most 0.4, which implies $\xi = 10$. The baseline calibration in King Plosser and Rebelo (1988), however, assumes an intermediate value of $\xi = 1$, which corresponds to a wage elasticity of 4.

The technological process ρ and σ_ϵ Following King, Plosser and Rebelo (1988), we set $\rho = 0.9$ and $\sigma_\epsilon^2 = 0.01^2$ which implies $\sigma_a^2 = (1-\rho^2)^{-1}\sigma_\epsilon^2 = 0.0227^2$.

Population moments The model moments can be computed analytically as follows. The variance-covariance matrix of the states Σ_s is

$$\Sigma_s = E\left[s_t s_t'\right] = G\Sigma_s G' + \Sigma_e$$

where Σ_e is the variance-covariance matrix of the disturbances. Following Hamilton (1994), p. 265, we can find solve for the elements of Σ_s as follows

$$vec(\Sigma_s) = [I - G \otimes G]^{-1} vec(\Sigma_e)$$

where I is the m^2-th order identity matrix. The autocovariance of z at lag j is

$$\Sigma_z = E\left[z_t z_t'\right] = HG^j \Sigma_s H'$$

The computations are in the matlab file 'rbcmodel.m', and the results for the baseline calibration are reproduced in table 1.[8] King et al. (1988) compare the population moments of the model to the moments of linearly detrended data. Kydland and Prescott (1982) follow a different approach and focus on HP-filtered data. They simulate artificial series from the model and estimate the population moments from the HP-filtered model simulated series. They compare these statistics to the sample HP-filtered statistics for the US. The computations are in the matlab file 'rbcmodel.m', and the results for the baseline calibration are reproduced in table 2.

[8]Note the difference in the output-hours correlation between our earlier established facts as well as Kydland and Prescott (1982) on the one hand and King, Plosser and Rebelo (1988) on the other hand. See the latter for a justification.

Table 1: **Business Cycle Moments using Linear Detrending**

variable x	$\sigma(x)$	$\frac{\sigma(x)}{\sigma(\hat{y})}$	$\rho(x,\hat{y})$	$\rho(x_t, x_{t-1})$
	King et al. (1988) (1988) linear detrended, sample 1950:Q1- 1986:Q4			
$\hat{y}$	5.62	1.00	1.00	0.96
$\hat{c}$	3.86	0.69	0.85	0.98
$\hat{i}$	7.61	1.35	0.60	0.93
$\hat{n}$	2.97	0.52	0.07	0.94
	Model, linear detrended			
$\hat{y}$	4.26	1.00	1.00	0.93
$\hat{c}$	2.73	0.64	0.82	0.99
$\hat{i}$	9.81	2.30	0.92	0.88
$\hat{n}$	2.05	0.48	0.79	0.86

Table 2: **Business Cycle Moments using HP-filter**

variable x	$\sigma(x)$	$\frac{\sigma(x)}{\sigma(\hat{y})}$	$\rho(x,\hat{y})$	$\rho(x_t, x_{t-1})$
	Kydland and Prescott (1982) HP-filtered, sample 1950:Q1- 1979:Q2			
$\hat{y}$	1.8	1.00	1.00	0.71
$\hat{c}$	1.3	0.72	0.74	
$\hat{i}$	5.1	2.83	0.71	
$\hat{n}$	2.0	1.11	0.85	
	Model, HP-filtered			
$\hat{y}$	2.07	1.00	1.00	0.68
$\hat{c}$	0.52	0.25	0.78	
$\hat{i}$	6.09	2.94	0.99	
$\hat{n}$	1.36	0.65	0.98	

3 Performance and Some Extensions of the RBC Model

3.1 Evaluation of the RBC Model

For a more recent overview and evaluation of the real business cycle literature, read King and Rebelo (1999) or Rebelo (2005).

EXERCISE: Substantiate the following criticisms of the basic RBC model.

1. The one sector neoclassical model is not capable of generating the degree of persistence we see in the data without introducing substantial serial correlation into the technology shocks, i.e. without high ρ.

2. For labor supply elasticities that are more in line with microeconometric evidence, the amplitude of the response to technology shocks is greatly diminished. For realistic labor elasticities, the RBC economy displays only small fluctuations in hours worked for relatively large fluctuations in productivity. As a consequence hours fluctuate too little relative to output.

3. The correlation between the real wage and output is much lower in the data then in the RBC model. (Find a real wage series yourself.)

4. Output growth is positively correlated in the data, but not in the model.

5. The Solow residual implies productivity variations that are implausibly large. The Solow residual often declines, suggesting that recessions are caused by technological regress. (Compute and plot the empirical Solow residual.)

3.2 Some Extensions of the Basic RBC Model

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