Additional Methods for Solving DSGE Models

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References

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Klein, P. (2000), 'Using the generalized schur form to solve a multivariate linear rational expectations model', *Journal of Economic Dynamics and Control* **24**(3), 482–498.

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We have solved the simple RBC model in King et al. (1988) applying the method of undetermined coefficients to the linearized system. The method is generalized by Uhlig 1997 and matlab software can be found at:

`http://www2.wiwi.hu-berlin.de/institute/wpol/html/toolkit.htm`

Here are two additional methods to find an approximate solution.

1 Linear Systems and the Schur (QZ) Decomposition

We can write the linearized system of stochastic difference equations as

$$AE_t\left[x_{t+1}\right] = Bx_t \tag{1}$$

where $x_t = [s_t, z_t]^T$ collects the state variables s_t and the controls z_t. Note that in the simple RBC model, we had chosen x_t such that A was invertible and $W = A^{-1}B$. The QZ decomposition does not require A to be invertible, which means that additional static (intratemporal) equilibrium conditions can be included among the dynamic relationships. We are looking for a solution of the form

$$\begin{aligned} s_{t+1} &= Gs_t + Fe_{t+1} \\ z_t &= Hs_t \end{aligned} \tag{2}$$

Consider the *complex generalized Schur/QZ decomposition*:

$$QAZ = S \text{ is upper triangular}$$
$$QBZ = T \text{ is upper triangular}$$

Moreover, $Q^HQ = Z^HZ = I$ where superscript H denotes the Hermitian transpose. Define the generalized eigenvalue λ_i as the ratio of the ith diagonal elements of T and S. If A is invertible, than λ_i is just the ith eigenvalue of W. If A is singular, some of its diagonal elements are zero and the corresponding λ_i is treated as infinite. It turns out there exists a real QZ decomposition for every ordering of λ_i. Let S and T be arranged in such a way that the stable (i.e. smaller than one) generalized eigenvalues come first and the unstable (exceeding one and infinite) come last. Correspondingly, define the auxiliary variables

$$x_t = Z\tilde{x}_t = Z \begin{bmatrix} x_t^s \\ x_t^u \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} x_t^s \\ x_t^u \end{bmatrix} \tag{3}$$

where x_t^s are the stable transformed variables and x_t^u are the unstable transformed variables. Hence

$$AZE_t\left[\tilde{x}_{t+1}\right] = BZ\tilde{x}_t \tag{4}$$

Premultiplying by Q gives a new system equivalent to (1)

$$SE_t \begin{bmatrix} x_{t+1}^s \\ x_{t+1}^u \end{bmatrix} = T \begin{bmatrix} x_t^s \\ x_t^u \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} E_t \begin{bmatrix} x_{t+1}^s \\ x_{t+1}^u \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} x_t^s \\ x_t^u \end{bmatrix} \tag{6}$$

where S_{11} and T_{22} are square and invertible by assumption. Hence, we can write

$$x_t^u = T_{22}^{-1} S_{22} E_t\left[x_{t+1}^u\right] \tag{7}$$

Furthermore, the generalized eigenvalues associated with S_{22} and T_{22} are all unstable by construction. Therefore, after solving forward, the solution for x_t^u will explode unless

$$x_t^u = 0$$

Given our solution for x_t^u, we have that

$$E_t\left[x_{t+1}^s\right] = S_{11}^{-1} T_{11} x_t^s \tag{8}$$

where $S_{11}^{-1}T_{11}$ is a stable matrix by construction. Defining $\tilde{e}_{t+1} = Z_{11}\left(x_{t+1}^s - E_t\left[x_{t+1}^s\right]\right)$ as the error in expectations and under the important assumption that Z_{11} is invertible we can write

$$\begin{bmatrix} x_{t+1}^s \\ x_{t+1}^u \end{bmatrix} = \begin{bmatrix} S_{11}^{-1}T_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_t^s \\ x_t^u \end{bmatrix} + \begin{bmatrix} Z_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{e}_{t+1} \\ 0 \end{bmatrix} \tag{9}$$

and after premultiplying by Z

$$x_{t+1} = Z \begin{bmatrix} S_{11}^{-1}T_{11} & 0 \\ 0 & 0 \end{bmatrix} Z^H x_t + \begin{bmatrix} \tilde{e}_{t+1} \\ 0 \end{bmatrix} \tag{10}$$

From the definition of the transformed variables we have that

$$\begin{aligned} s_{t+1} &= \begin{bmatrix} Z_{11} & Z_{12} \end{bmatrix} \begin{bmatrix} x_{t+1}^s \\ x_{t+1}^u \end{bmatrix} \\ z_t &= \begin{bmatrix} Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} x_t^s \\ x_t^u \end{bmatrix} \end{aligned}$$

which leads to

$$\begin{aligned} s_{t+1} &= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}s_t + \tilde{e}_{t+1} \\ z_t &= Z_{21}Z_{11}^{-1}s_t \end{aligned}$$

The final step is to pin down the expectational error $\tilde{e}_{t+1}$, which is done by theoretical motivations:

$$\tilde{e}_{t+1} = Fe_{t+1}$$

The solution to the King et al. (1988) model is computed with the QZ decomposition in the Matlab program **rbcmodel2.m** using the code provided by Klein (2000) (solab.m).

2 The Optimal Linear Regulator

Consider the following linear-quadratic control problem:

$$\begin{aligned} \max_{\{z_t\}_{t=0}^{\infty}} \quad & E_0 \sum_{t=0}^{\infty} \beta^t \left(\tilde{s}_t^T R \tilde{s}_t + z_t^T Q z_t + \tilde{s}_t^T W z_t + z_t^T W^T \tilde{s}_t \right) && (11) \\ \text{s.t. } \tilde{s}_{t+1} \; = \; & A\tilde{s}_t + B z_t + e_{t+1} && (12) \\ & \tilde{s}_0 \text{ given} \end{aligned}$$

where superscript T denotes the transpose, R and Q are symmetric negative semidefinite matrices and e_{t+1} is a vector of mean zero iid shocks with $E\left[e_{t+1} e_{t+1}^T\right] = \Omega$. Assume the value function is given by

$$V(\tilde{s}) = \tilde{s}^T P \tilde{s} + d$$

where P is a symmetric negative semidefinite matrix. Dropping the time subscripts, the value function can be written as

$$\tilde{s}^T P \tilde{s} + d \; = \; \max_z \left(\tilde{s}^T R \tilde{s} + z^T Q z + \tilde{s}^T W z + z^T W^T \tilde{s} + \beta E\left[(A\tilde{s} + Bz + e')^T P (A\tilde{s} + Bz + e') + d \right] \right)$$

The first order condition is

$$\begin{aligned} z \; : \; & z^T Q + z^T Q^T + 2\tilde{s}^T W + 2\beta E (A\tilde{s} + Bz + e')^T P B = 0 \\ \Leftrightarrow \quad & Q z + W^T \tilde{s} + \beta E B^T P (A\tilde{s} + Bz + e') = 0 \\ \Leftrightarrow \quad & z = -\left(Q + \beta B^T P B\right)^{-1} \left(W^T + \beta B^T P A\right) \tilde{s} \end{aligned}$$

Plugging the solution for z into (12):

$$\tilde{s}' \; = \; \Big(A - B \left(Q + \beta B^T P B\right)^{-1} \left(W^T + \beta B^T P A\right) \Big) \tilde{s} + e'$$

We still don't know P, however, plugging our solution for z into the value function, we get that P must solve

$$\begin{aligned} \tilde{s}^T P \tilde{s} + d \; = \; & \tilde{s}^T R \tilde{s} + \tilde{s}^T F^T Q F \tilde{s} - \tilde{s}^T W F \tilde{s} - \tilde{s}^T F^T W^T \tilde{s} \\ & + \beta E\left[((A - BF)\tilde{s} + e')^T P ((A - BF)\tilde{s} + e') + d \right] \\ = \; & \tilde{s}^T \left(R + F^T Q F - W F - F^T W^T + \beta (A - BF)^T P (A - BF) \right) \tilde{s} + E\beta\left[(e')^T P e' \right] + \beta d \end{aligned}$$

where $F = \left(Q + \beta B^T PB\right)^{-1}\left(W^T + \beta B^T PA\right)$. Using the method of undetermined coefficients, we have that P and d must solve

$$\begin{aligned} P &= R + F^T QF - WF - F^T W^T + \beta(A - BF)^T P(A - BF) \\ d &= \beta d + E\beta\left[(e')^T Pe'\right] \end{aligned}$$

and after some tedious algebra

$$P = R + \beta A^T PA + \left(\beta A^T PB + W\right)\left(Q + B^T PB\right)^{-1}\left(\beta B^T PA + W^T\right) \tag{13}$$

$$d = \frac{\beta}{1-\beta} tr\left(P\Omega\right) \tag{14}$$

Equation (13) cannot be solved analytically, but is under some regularity conditions straigthforward to solve numerically by iterating on the *matrix Ricatti difference equation*. Start with an initial guess P_0 and then iterate

$$P_{j+1} = R + \beta A^T P_j A + \left(\beta A^T PB + W\right)\left(Q + B^T P_j B\right)^{-1}\left(\beta B^T P_j A + W^T\right)$$

until convergence. After finding P we can use (14) to solve for d, although it enters only the value function and not the policy functions.

Now why is all this useful? It is useful because we can approximate the planner's problem in our RBC model as a linear-quadratic problem such as (11). Recall the recursive formulation of the planning problem in the simple RBC model

$$\begin{aligned} v(k,a) &= \max_{k',N} u(\bar{A}e^a k^{1-\alpha} N^\alpha + i, 1 - N) + \beta E\left[v(k',a') \mid a\right] \\ \text{s.t. } i &= \gamma_x k' - (1-\delta)k \\ a' &= \rho a + \epsilon' \end{aligned}$$

Let $\tilde{s} = [1, s]^T = [1, k, a]^T$ and let $z = [N, i]$. Consider a second order Taylor approximation of u around the deterministic steady state:

$$\begin{aligned} u_t &\approx \bar{u} + u_s(s_t - \bar{s}) + u_z(z_t - \bar{z}) + \frac{1}{2}(s_t - \bar{s})^T u_{ss}(s_t - \bar{s}) + \frac{1}{2}(z_t - \bar{z})^T u_{zz}(z_t - \bar{z}) \\ &\quad + \frac{1}{2}(z_t - \bar{z})^T u_{zs}(s_t - \bar{s}) + \frac{1}{2}(s_t - \bar{s})^T u_{sz}(z_t - \bar{z}) \\ &\approx \begin{bmatrix} 1 & s_t^T \end{bmatrix} \begin{bmatrix} \bar{u} - u_s\bar{s} - u_z\bar{z} + \frac{1}{2}\left(\bar{s}^T u_{ss}\bar{s} + \bar{z}^T u_{zz}\bar{z} + \bar{z}^T u_{zs}\bar{s} + \bar{s}^T u_{sz}\bar{z}\right) & \frac{1}{2}(u_s - \bar{s}^T u_{ss} - \bar{z}^T u_{zs}) \\ \frac{1}{2}(u_s^T - u_{ss}\bar{s} - u_{sz}\bar{z}) & \frac{1}{2}u_{ss} \end{bmatrix} \begin{bmatrix} 1 \\ s_t \end{bmatrix} \\ &\quad + \frac{1}{2}\begin{bmatrix} 1 & s_t^T \end{bmatrix} \begin{bmatrix} u_z - \bar{z}^T u_{zz} - \bar{s}^T u_{sz} \\ u_{sz} \end{bmatrix} z_t + \frac{1}{2} z_t^T \begin{bmatrix} u_z^T - u_{zz}\bar{z} - u_{zs}\bar{s} & u_{zs} \end{bmatrix} \begin{bmatrix} 1 \\ s_t \end{bmatrix} + \frac{1}{2} z_t^T u_{zz} z_t \\ &\approx \tilde{s}_t^T R \tilde{s}_t + \tilde{s}_t^T W z_t + z_t^T W^T \tilde{s}_t + z_t^T Q z_t \end{aligned}$$

The solution to the King et al. (1988) model is computed using the optimal linear regulator in the Matlab program **rbcmodel3.m**. Note that in the matlab program, the approximation is linear instead of loglinear. You can read about the optimal linear regulator in Ljunqvist & Sargent (2004).